Note Title 17/06/2015

Large deviations

for invariant measures

for 2-D Stochastic

Navier Stokes Equations

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I would like
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interesting workshop.

Definition 1.

A family $(\mu_\varepsilon)_{\varepsilon>0}$ of measures on a Polish space X satisfies the LDP with rate (action) function

$$I : X \longrightarrow [0,\infty]$$

iff

(1) $\forall r \geq 0 \quad K_r := \{x \in X : I(x) \leq r\}$ is a compact subset of X

(2) $\forall F \subset X$ closed

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mu^\varepsilon(F) \leq -I(F)$$

(3) $\forall G \subset X$ open

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mu^\varepsilon(G) \geq -I(G)$$

where

$$I(A) := \inf\{I(x) : x \in A\}$$

Remarks. $(2) \Leftrightarrow (2^*)$

$(3) \Leftrightarrow (3^*)$

(2^*) $\forall s > 0 \ \forall \delta > 0 \ \forall \gamma \in (0, s)$

$\exists \varepsilon_0 > 0 : \forall \ \varepsilon \in (0, \varepsilon_0)$

$$\varepsilon \log \mu_\varepsilon(\{x \in X : d(x, K_s) \geq \delta\}) \leq -s + \gamma$$

(3^*) $\forall x_0 \in X \ \forall \delta > 0 \ \forall \gamma > 0$

$\exists \varepsilon_0 > 0 : \ \forall \varepsilon \in (0, \varepsilon_0)$

$$\varepsilon \log \mu_\varepsilon (B(x_0, \delta)) \geq -I(x_0) - \gamma$$

Example 2.

E – separable Banach space

μ – symmetric gaussian measure on E

$K = K_\mu \subset E$
Reproducing Kernel Hilbert space of μ
(Cameron Martin space)

Then

$$\mu = \mathrm{Law}\left(\sum_{j=1}^{\infty} \beta_j e_j\right)$$

$\{e_j\}$ ONB of K

(β_j) i.i.d. $N(0,1)$ random variables

Put

$$\mu_\varepsilon := \mathrm{Law}\left(\sqrt{\varepsilon}\sum_j \beta_j e_j\right)$$

Then $(\mu_\varepsilon)_{\varepsilon>0}$ satisfies

LDP with rate

function

$$I(x) = \begin{cases} \frac{1}{2}|x|_K^2, & x \in K \\ \infty, & x \in E \setminus K. \end{cases}$$

Example 3. $T > 0$

$$\mathcal{E} = {}_0C([0,T]; E)$$

$$= \{ u \in C([0,T], E) : u(0) = 0 \}$$

$W = (W_t)_{t \geq 0}$ — E valued

Wiener process

K = RKHS of $\mathrm{Law}(W_1)$

Put

$$\mu_\varepsilon = \mathrm{Law}(\sqrt{\varepsilon}\, W_\cdot)$$

on $\mathcal{E}$

Then (μ_ε) satisfy LDP with

$$I(u) = \begin{cases} \frac{1}{2} \int_0^T |\dot{u}(t)|_K^2 \, dt, & (*) \\ \infty & \end{cases}$$

$(*)$ if $u \in {}_0H^{1,2}(0,T; K)$.

Lemma 4 (Continuity Lemma of Freidlin Wentzell and Varadhan)

If $\varphi : E \to F$ continuous

$\{\mu_\varepsilon\}$ satisfy LDP on E

with $I : E \to [0, \infty]$

then $\nu_\varepsilon = \mu_\varepsilon \circ \varphi$, $A \mapsto \mu_\varepsilon(\varphi^{-1}(A))$

satisfy LDP with

$$J(y) := \inf_{y \in F} \{ I(x) : \varphi(x) = y \}$$

Note: $J(y) < \infty$ iff $\exists x \in \varphi^{-1}(y)$ $I(x) < \infty$

Example 5. Navier - Stokes Equations in 2-Dim domain $\mathcal{O}$ (Dirichlet or periodic boundary conditions)

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla) u + \nabla p = \sqrt{\varepsilon}\, \dot{f} \\ \operatorname{div} u = 0 \\ u(0, \cdot) = u_0 \end{cases}$$

f - noise (temporality uncorrelated, spatially correlated or not)

Put

$$H = \{ u \in \mathbb{L}^2(\mathcal{O}) : \quad \operatorname{div} u = 0$$
$$u \cdot \nu \big|_{\partial\mathcal{O}} = 0 \quad \}$$

(or u is periodic)

$$V = H \cap \mathbb{H}_0^{1,2}(\mathcal{O})$$

Then

$$\begin{cases} du + [Au + B(u,u)]\,dt \\ \qquad = \sqrt{\varepsilon}\, dW(t) \\ u(0) = x \in H \end{cases}$$

$$W(t) = \sum_{j=1}^{\infty} w_j(t)\, e_j$$

$\{e_j\}$ ONB of $K \subset H$

$\{w_j\}$ i.i.d. BM's

Here

$$\Pi : \mathbb{L}^2(\mathcal{O}) \longrightarrow H$$

orthogonal projection

$$A = \Pi \circ (-\Delta)$$

$$D(A) = V \cap \mathbb{H}^{2,2}(\mathcal{O})$$

$$B(u,u) = \Pi((u \cdot \nabla)u)$$

Note that

$$\Pi(\nabla p) = 0$$

Under some assumptions

e.g. $\exists\, \delta < \frac{1}{2}$:

$$A^{-\delta} : K \longrightarrow H \cap \mathbb{L}^4$$

is γ-radonifying,

there exists a unique

solution u_x^ε (with

continuous H-valued

trajectories).

Moreover, by the

WFV Lemma,

If

$$\mu_\varepsilon = \mathrm{Law}\,(u_x^\varepsilon)$$

on $C([0,T], H) \cap L^4(0,T; L^4)$

then $(\mu_\varepsilon)_{\varepsilon>0}$ satisfy LDP

with action functional

$$S_T(u) = S_{0,T}(u)$$
$$= \frac{1}{2}\int_0^T |u' + Au + B(u,u)|_K^2 \, dt$$

(uniformly for x in balls in H)

Recall

$$\begin{cases} du_x^\varepsilon + (Au_x^\varepsilon + B(u_x^\varepsilon, u_x^\varepsilon))dt \\ \qquad = \sqrt{\varepsilon}\, dW \\ u_x^\varepsilon(0) = x \end{cases}$$

If $x, y \in H$ we put

$$U_{0,T}(x,y) := \inf \frac{1}{2} \int_0^T |u' + Au + B(u,u)|_K^2 \, dt$$

over u: $u(0) = x$, $u(T) = y$

Fix $x \in H$, $T > 0$. Put

$\nu^\varepsilon = \text{Law}\left(u^\varepsilon_x(T)\right)$

$\uparrow$ a Borel prob. measure on H

By FWV Lemma

$\{\nu^\varepsilon\}_{\varepsilon>0}$ satisfy LDP on H

with action functional

$$y \longmapsto \mathcal{U}_{0,T}(x, y)$$

Since the unforced NSE

$$\frac{\partial u}{\partial t} + Au + B(u,u) = 0$$

has only one stationary solution (which is asymptotically stable) we put

$$U(x) = \inf_{T>0} U_{0,T}(0,x)$$

Theorem 6 (Brz + Cerrai + Freidlin, PTRF to appear)

If $K \cong D(A^{\alpha/2})$, $0<\alpha<\frac{1}{2}$

then

$$u(x) = \frac{1}{2} \inf \int_{-\infty}^{0} |u' + Au + B(u,u)|_K^2 \, dt$$

$$u(-\infty) = 0, \quad u(0) = x$$

Moreover, for the periodic b.c. the above holds also for $0 < \alpha < 3/2$.

Proof. Somehow related to the proof of asymptotic compactness of the random dynamical system generated by 2D stochastic NSE's (Brz + Li, TAMS, 2006)

For many reasons we are interested in the space-time noise, i.e. noise spatially uncorellated, i.e

$$K = H.$$

Approximation: Fix

$$0 < \alpha < \frac{1}{2}$$

Put $\delta > 0$.

$$K_\delta = D(I + \delta A^{\alpha/2})$$

Corresponding quasipotential

If $W(t), t \geq 0$ is H-cylindrical Wiener process

$$W(t) = \sum_j w_j(t)\, e_j$$

$\{e_j\}$ ONB of H

$\{w_j\}$ i.i.d. real BM's

then our Stochastic NSE's

$$\begin{cases} du + (Au + B(u,u))\,dt \\ \quad = \sqrt{\varepsilon}\,(I + \delta A^{\alpha/2})^{-1}\, dW \\ u(0) = x \in H \end{cases}$$

This problem is well posed for $\delta > 0$ (under one extra technical assumption mentioned earlier).

$$Q_\delta(x) = \frac{1}{2} \inf \int_{-\infty}^{0} \left| \left(I + \delta A^{\alpha/2} \right) \left(u' + Au + B(u,u) \right) \right|_H^2 dt$$

over u: $u(-\infty) = 0$, $u(0) = x$

Proposition 7 (Brz + Cerrai + Freidlin)

$U_\delta(x) < \infty$ iff $x \in D(A^{\alpha/2})$

Theorem 8 (—"—)

(i) Level sets of U_δ are compact. Hence U_δ is semi-continuous

(.) $U_\delta(x) \to U(x)$

as $\delta \searrow 0$

for $x \in D(A^{\alpha/2})$

where

$$U(x) = \frac{1}{2} \inf \int_{-\infty}^{0} |u' + Au + B(u,u)|^2 ds$$

over $u(-\infty) = 0$
$u(0) = x$

Example 9.

NSE's with periodic b.c.

$$\langle Au, B(u,u) \rangle = 0$$

Then (BCF)

$$U(x) = \begin{cases} \frac{1}{2} \|x\|_V^2, & x \in V \\ \infty \end{cases}$$

where $V = H \cap H^{1,2}_{per}$

$= D(A^{1/2})$

$$\|\phi\|_V^2 = \int_O |\nabla \phi(\xi)|^2 \, d\xi$$

Example 10 (Brz + Cerrai, in preparation)

If $\bar{U}(x) = \frac{1}{2} \inf \int_{-\infty}^0 |u' + Au + B(u,u)|_{V'}^2 \, dt$

then
$$\tilde{U}(x) = \begin{cases} \frac{1}{2} |x|_H^2, & x \in H \\ \infty, & x \in V' \setminus H \end{cases}$$

Consider stochastic Stokes equation

$$du + Au\,dt = \sqrt{\varepsilon}\,dW$$

2- sided Wiener process

Stationary solution

$$u^{\varepsilon}(t) = \sqrt{\varepsilon}\int_{-\infty}^{t} e^{-(t-s)A}\,dW(s)$$

$$u^{\varepsilon}(0) = \sqrt{\varepsilon}\int_{-\infty}^{0} e^{sA}\,dW(s)$$

Then measures

$$\nu^{\varepsilon} = \mathrm{Law}\left(u^{\varepsilon}(0)\right)$$

on H

satisfy LDP with

action functional

$$x \mapsto \frac{1}{2} \inf \int_{-\infty}^{0} |u' + Au(s)|^2 \, ds$$

over u: $u(-\infty) = 0$

$u(0) = x$

$=: \tilde{\tilde{u}}(x)$

What about NSE's ?

Let ν^ε be the unique invariant measure on H of Markov process corresponding to

$$du^\varepsilon + (Au^\varepsilon + B(u^\varepsilon, u^\varepsilon))\, dt = \sqrt{\varepsilon}\, dw$$

where $w(t)$, $t \geq 0$ is a cylindrical Wiener process on

$K = D(A^{\alpha/2})$, $\alpha > 0$

Then $(\mu^\varepsilon)_{\varepsilon>0}$ satisfy LDP

with action function

$$Q_K(x) = \frac{1}{2} \inf \int_{-\infty}^{0} |u' + Au + B(u,u)|_K^2 \, dt$$

$$u(0) = x, \quad u(-\infty) = 0$$

Partial answer.

Yes for NSE with periodic b.c. and $1 < \alpha < 3/2$

Ingredients of the proof.

(1) $$2\,Q_K(x) = \inf_{\substack{u,\,T \\ u(-T)=0 \\ u(0)=x}} \int_{-T}^{0} |u' + Au + Bu|^2 \, dt$$

(2) Given $x \in H$, $T > 0$

Law (u_x^{ε}) satisfies LDP

on $C([0,T],H) \cap L^4(0,T;L^4)$

with action functional

$$\inf_{u(-T)=0,\ u(0)=x} \int_{-T}^{0} |u' + Au + B(u,u)|_K^2 \, dt$$

(3) Exponential estimates

$\forall s > 0 \ \exists \varepsilon_0 > 0 \ \exists R_s > 0$

$$\nu^{\varepsilon}\left(B_H^c (0, R_s) \right) \leq e^{-s/\varepsilon}$$

$0 < \varepsilon \leq \varepsilon_0$

Follows from exponential estimates on solutions (Bn + Perrat 2000)

(4) Lower bounds $\forall \bar{x} \in H$

$\forall \delta > 0 \ \forall \gamma > 0 \quad \exists \varepsilon_0 > 0$:

$$\nu_\varepsilon \left(B_H (\bar{x}, \delta) \right) \geq e^{-\frac{U(\bar{x}) + \gamma}{\varepsilon}}$$

for $\quad 0 < \varepsilon \leq \varepsilon_0$

(5) Upper bounds

$$\forall \delta > 0 \quad \forall \gamma > 0 \quad \forall s \geq 0 \quad \exists \varepsilon_0 > 0$$

$$\nu_\varepsilon\left(\{x \in H : d(x, K_s) \geq \delta\}\right) \geq e^{-\frac{s-\gamma}{\varepsilon}}, \quad 0 < \varepsilon < \varepsilon_0$$

(6) Uniqueness (Ferrario 1999) of an inv. measure (and ergodicity)

$$\int_H \varphi(x)\, \nu_\varepsilon(dx) = \int_H E\left(\varphi\left(u_x^\varepsilon(t)\right)\right) \nu^\varepsilon(dx)$$

$$\int \varphi(x)\, \nu_\varepsilon(dx) = \lim_{T \to \infty} \frac{1}{T} \int_0^T E\left(\varphi\left(u_0^\varepsilon(t)\right)\right) dt$$

Future developments

(1) General domains

(2) Rougher noise

(3) Degenerate noise

(4) NSE's in unbounded domains

(5) Multiplicative noise

(6) other problems, e.g. Landau-Lifshitz-Gilbert equations (2 stationary locally asymptotically stable solutions)

with M. Mariani and Liang Li

Thanks for

your attention